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ON EDGE-DISJOINT BRANCHINGS

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Edmonds has given a complicated algorithmic proof of a theorem characterizing directed graphs that contain k edge-disjoint branchings having specified root sets. Tarjan has described a conceptually simple and good algorithm for finding such branchings when they exist. Tarjan's algorithm is based on a lemma implicit in Edmonds' results. A simple direct proof of this lemma is given, thereby providing a simpler proof of Edmonds' theorem and a simpler proof that Tarjan's algorithm works.

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ON EDGE-DISJOINT BRANCHINGS

by

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1. Introduction. In [1] Edmonds has given a proof of a theorem (Theorem 3.1 below) characterizing those directed graphs that contain k mutually edge-disjoint branchings (spanning arborescences) having specified root sets. His proof is based on a complicated algorithm for constructing such branchings when they exist. While it is not known whether this algorithm is good (runs in polynomial time), Tarjan has described a conceptually simple and good algorithm for finding k mutually edge-disjoint branchings, when they exist [4]. Tarjan's algorithm is based on a lemma (Lemma 2 of [4]; slightly generalized below as Theorem 3.2) and network flow routines. Tarjan's proof of this lemma invokes Edmonds' theorem and algorithm; indeed, as Tarjan notes, his lemma is implicit in Edmonds' results. This poses the problem, pointed out by Tarjan in [4], of finding a simple direct proof of his lemma, one that avoids invoking Edmonds' theorem and its complicated algorithmic proof. The purpose of this note is to give such a proof, thereby providing a simpler proof of Edmonds' theorem and a simpler proof that Tarjan's algorithm works.

2. Definitions and notation. A directed graph $G = [V,E]$ consists of a finite set of vertices V and a finite set of edges E such that each edge $e \in E$ has a head $h(e) \in V$ and a tail $t(e) \in V$. We sometimes denote an edge e by the ordered pair $(t(e),h(e))$, even though there may be multiple edges in G having the same tail and head. A subgraph $G' = [V',E']$ of G is a directed graph having vertex-set $V' \subseteq V$ and edge-set $E' \subseteq E$ such that for all $e \in E'$, we have $t(e) \in V'$ and $h(e) \in V'$. A directed path in G from $u \in V$ to $v \in V$ is a sequence $u = t(e_1), e_1, h(e_1) = t(e_2), e_2, \ldots, h(e_{n-1}) = t(e_n), e_n, h(e_n) = v$, composed alternately of vertices and edges of G. A vertex u is itself a directed path from u to u having no edges.

For any non-empty subset R of vertices of G, a branching B of G, rooted at R, is a subgraph of G such that for every vertex v of G, there is precisely one directed path in B from a vertex in R to v.

For any $X \subseteq V$, let

$$\delta_G^+(X) = \{e \in E\colon t(e) \in X \text{ and } h(e) \in \bar{X} = V - X\},$$

$$\delta_G^-(X) = \{e \in E\colon h(e) \in X \text{ and } t(e) \in \bar{X} = V - X\}.$$

For $X \subseteq V$, $Y \subseteq V$, we use the notation

$$(X,Y) = \{e \in E\colon t(e) \in X,\ h(e) \in Y\}.$$

Thus $\delta_G^+(X) = (X,\bar{X})$, $\delta_G^-(X) = (\bar{X},X)$. If X is a non-empty proper subset of V, the set $\delta_G^+(X) = \delta_G^-(\bar{X}) = (X,\bar{X})$ is a cut in G; it separates any vertex $x \in X$ from any vertex $y \in \bar{X}$; i.e., any directed path in G from $x \in X$ to $y \in \bar{X}$ contains at least one edge of $\delta_G^+(X)$.

If S is a set, we let $|S|$ denote the cardinality of S. As above, we use the symbol "$\subseteq$" for set inclusion; henceforth we use "$\subset$" for proper inclusion.

3. Main theorems. Let $G = [V,E]$ be a directed graph with designated non-empty root-sets $R_1,R_2,\ldots,R_k$, and suppose that G contains k mutually edge-disjoint branchings $B_1,B_2,\ldots,B_k$, where B_i is rooted at R_i, $i = 1,2,\ldots,k$. Then it is clear that for every proper subset X of V, we must have

$$|\delta_G^+(X)| \geq |\{i\colon 1 \leq i \leq k \text{ and } R_i \subseteq X\}|.$$

Theorem 3.1 (Edmonds). *For any directed graph* $G = [V,E]$ *and any sets* R_i, $\emptyset \neq R_i \subseteq V$, $1 \leq i \leq k$, *there exist mutually edge-disjoint branchings* B_i, $1 \leq i \leq k$, *rooted respectively at* R_i, *if and only if, for every proper subset* X *of* V, *we have*

$$|\delta_G^+(X)| \geq |\{i\colon 1 \leq i \leq k \text{ and } R_i \subseteq X\}|. \tag{3.1}$$

Theorem 3.2. *Suppose given a directed graph* $G = [V,E]$ *and a class of non-empty subsets* $R_1, R_2, \ldots, R_k$ *of* V *such that (3.1) holds for all* $X \neq V$. *Let* $B_1 = [V_1, E_1]$ *be a subgraph of* G *such that* $R_1 \subseteq V_1$ *and on the subgraph* $G' = [V, E-E_1]$ *we have*

$$|\delta_{G'}^+(X)| \geq |\{i\colon 2 \leq i \leq k \text{ and } R_i \subseteq X\}|, \quad \text{all } X \neq V. \tag{3.2}$$

Then if $V_1 \neq V$, *there is an edge* $e^* \in \delta_G^+(V_1)$ *and for all* $X \neq V$,

$$e^* \in \delta_{G'}^+(X) \Rightarrow |\delta_{G'}^+(X)| \geq |\{i\colon 2 \leq i \leq k \text{ and } R_i \subseteq X\}| + 1. \tag{3.3}$$

In applying Theorem 3.2 and Tarjan's algorithm to prove Theorem 3.1, the subgraph B_1 of Theorem 3.2 would be taken to be a branching rooted at R_1 of some subgraph of G. In this instance, Theorem 3.2 reduces to Lemma 2 of [4].

4. Proof of Theorem 3.2. We begin the proof of Theorem 3.2 with some preliminary lemmas. It is convenient first to extend G by adding a "source" vertex s and vertices $r_1, r_2, \ldots, r_k$ corresponding to the root-sets $R_1, R_2, \ldots, R_k$. We also add the edges (s, r_i), together with the sets of edges

(r_i, R_i), $i = 1,2,\ldots,k$, thereby obtaining an enlarged directed graph $H = [N,A]$ containing G as a subgraph. Note that all edges joining $N-V$ and V in H are directed into V, i.e. $(V, N-V) = \emptyset$. Corresponding to the subgraph $B_1 = [V_1, E_1]$ of G there is an "s-rooted subgraph" $\hat{B}_1$ of H having vertex-set $V_1 \cup \{s, r_1\}$ and edge-set $A_1 = E_1 \cup (s, r_1) \cup (r_1, R_1)$; hence, corresponding to the subgraph $G' = [V, E-E_1]$ of G there is the subgraph $H' = [N, A-A_1]$ of H.

Lemma 4.1. *Condition (3.1) implies that there are at least k mutually edge-disjoint directed paths in H from s to v, for all $v \in V$.*

Proof. By the max-flow min-cut theorem and the integrity theorem for network flows [2], it suffices to show that if $(S, N-S)$ is a cut in H separating s from v, then $|(S, N-S)| = |\delta^+_H(S)| \geq k$. Let $(S, N-S)$ be a cut in H with $s \in S$, $v \in N-S = \bar{S}$. Let $R = \{r_1, r_2, \ldots, r_k\}$. We may partition $(S, \bar{S})$ as follows:

$$(4.1) \qquad (S,\bar{S}) = (s, R \cap \bar{S}) \cup (R \cap S, V \cap \bar{S}) \cup (V \cap S, V \cap \bar{S}).$$

Suppose $R_i \not\subseteq S$. Then either $r_i \notin S$, in which case $(s, r_i) \in (s, R \cap \bar{S})$, or $r_i \in S$ and there is a vertex $u \in R_i \cap \bar{S}$, in which case $(r_i, u) \in (R \cap S, V \cap \bar{S})$. Thus

$$(4.2) \qquad |(s, R \cap \bar{S}) \cup (R \cap S, V \cap \bar{S})| \geq |\{i: 1 \leq i \leq k \text{ and } R_i \not\subseteq S\}|.$$

Since $v \in V \cap \bar{S}$, we have $V \cap S \subset V$, and hence condition (3.1) implies

$$(4.3) \qquad |(V \cap S, V \cap \bar{S})| \geq |\{i: 1 \leq i \leq k \text{ and } R_i \subseteq S\}|.$$

It follows from (4.1), (4.2), and (4.3) that $|(S,\bar{S})| \geq k$, as was to be shown.

A similar proof establishes

Lemma 4.2. Condition (3.2) implies that there are at least $k-1$ mutually edge-disjoint directed paths in H' from s to v, for all $v \in V$.

We next state two lemmas that are valid for any directed graph with "source" s. (Later on they will be applied to the directed graph H'.) While these lemmas can be found in a recent paper by Lovász [3], they are consequences of well-known results in network flow theory. In particular, the second of the two (Lemma 4.4 below) is stated explicitly in [2, Chap. I]. We describe these lemmas as in [3], using the following definition. In a directed graph $H = [N,A]$ with "source" $s \in N$, let $m(s,x)$ denote the maximum number of mutually edge-disjoint directed paths from s to x, for $x \in N-\{s\}$. Say that a set $X \subseteq N-\{s\}$ is regular with core x if $x \in X$ and $m(s,x) = \delta_H^-(X)$. (In other words, X is the "sink" set of a minimum cut $(\bar{X},X)$ separating $s \in \bar{X}$ from $x \in X$ in H.)

Lemma 4.3. If X and Y are regular sets with cores x and y, respectively, and if $x \in Y$, then $X \cap Y$, $X \cup Y$ are regular with cores x, y, respectively.

Lemma 4.4. For each vertex $x \neq s$, there is a regular set T_x with core x such that whenever X is a regular set with core x, then $T_x \subseteq X$.

We continue with the proof of Theorem 3.2. Suppose that (3.1) and (3.2) hold, but that (3.3) does not hold. Let e_j, $j \in J$, be an enumeration of the edges of G comprising the set $\delta_G^+(V_1)$. Thus for each $e_j \in \delta_G^+(V_1)$ there is a set $S_j \neq V$ such that $e_j \in \delta_{G'}^+(S_j)$ and

$$|\delta^+_{G'}(S_j)| < |\{i\colon 2 \le i \le k \text{ and } R_i \subseteq S_j\}| + 1.$$

Combined with (3.2), this yields

$$|\delta^+_{G'}(S_j)| = |\{i\colon 2 \le i \le k \text{ and } R_i \subseteq S_j\}| \tag{4.4}$$

for all $j \in J$.

We want to work with the enlarged directed graphs H and H', rather than G and G'. Hence we define

$$T_j = S_j \cup \{s\} \cup \{r_i\colon R_i \subseteq S_j\}. \tag{4.5}$$

It follows that

$$|\delta^+_{H'}(T_j)| = k-1, \quad j \in J. \tag{4.6}$$

To see this, note first that if $R_i \not\subseteq S_j$, then $r_i \notin T_j$, and hence $(s,r_i) \in \delta^+_{H'}(T_j)$, whereas no edge e of H' with tail $t(e) = r_i$ belongs to $\delta^+_{H'}(T_j)$. On the other hand, if $R_i \subseteq S_j$, then no edge incident to r_i belongs to $\delta^+_{H'}(T_j)$. Thus

$$|\delta^+_{H'}(T_j)| = |\delta^+_{G'}(S_j)| + |\{i\colon 2 \le i \le k \text{ and } R_i \not\subseteq S_j\}|.$$

By (4.4), we have

$$\begin{aligned} |\delta^+_{H'}(T_j)| &= |\{i\colon 2 \le i \le k \text{ and } R_i \subseteq S_j| + |\{i\colon 2 \le i \le k \text{ and } R_i \not\subseteq S_j\}| \\ &= k-1, \end{aligned}$$

verifying (4.6).

Since $s \in T_j$ and $h(e_j) \in \bar{T}_j = N - T_j$, the set $\delta^+_{H'}(T_j)$ is a cut in H' of size $k-1$ separating s and $h(e_j)$, for all $j \in J$. Lemma 4.2 thus implies that $\bar{T}_j$ is regular with core $h(e_j)$. Using Lemma 4.4, we may assume that $\bar{T}_j$ is minimal with core $h(e_j)$. (Note that with this assumption, we still have $e_j \in \delta^-_{H'}(\bar{T}_j)$.)

Lemma 4.5. *Let the sets* $\bar{T}_j$ *be minimal regular sets in* H' *with cores* $h(e_j)$, $j \in J$. *Suppose that* $j, \ell \in J$ *with* $h(e_\ell) \in \bar{T}_j$. *Then* $\bar{T}_\ell \subseteq \bar{T}_j$. *If* $\bar{T}_\ell \subset \bar{T}_j$, *then* $h(e_j) \notin \bar{T}_\ell$.

Lemma 4.5 follows from Lemma 4.3, since if $\bar{T}_\ell$ and $\bar{T}_j$ are regular with cores $h(e_\ell)$ and $h(e_j)$, respectively, and if $h(e_\ell) \in \bar{T}_j$, then $\bar{T}_\ell \cap \bar{T}_j$ is regular with core $h(e_\ell)$. Since $\bar{T}_\ell$ is minimal with respect to this property, we have $\bar{T}_\ell \subseteq \bar{T}_\ell \cap \bar{T}_j$, and hence $\bar{T}_\ell \subseteq \bar{T}_j$. If this inclusion is proper and if $h(e_j) \in \bar{T}_\ell$, then $\bar{T}_\ell$ would be regular with core $h(e_j)$, contradicting the minimality of $\bar{T}_j$. Thus if $\bar{T}_\ell \subset \bar{T}_j$, then $h(e_j) \notin \bar{T}_\ell$.

We apply Lemma 4.5 repeatedly to prove the next lemma.

Lemma 4.6. *Let the sets* $\bar{T}_j$ *be minimal regular sets in* H' *with cores* $h(e_j)$, $j \in J$. *There is at least one* $j^* \in J$ *such that if* $j \in J$ *and* $h(e_j) \in \bar{T}_{j^*}$, *then* $\bar{T}_j = \bar{T}_{j^*}$.

To prove this lemma, select any $j_0 \in J$. Define $J_0 = \{j \in J: h(e_j) \in \bar{T}_{j_0}\}$. If for all $j \in J_0$, we have $\bar{T}_j = \bar{T}_{j_0}$, then take $j^* = j_0$. Otherwise, there is a $j_1 \in J_0$ such that $\bar{T}_{j_1} \neq \bar{T}_{j_0}$, in which case Lemma 4.5 asserts that $\bar{T}_{j_1} \subset \bar{T}_{j_0}$, $h(e_{j_0}) \notin \bar{T}_{j_1}$. Define $J_1 = \{j \in J: h(e_j) \in \bar{T}_{j_1}\}$. If for all $j \in J_1$, we have $\bar{T}_j = \bar{T}_{j_1}$, then take $j^* = j_1$. Otherwise, there is a $j_2 \in J_1$ such that $\bar{T}_{j_2} \neq \bar{T}_{j_1}$, in which case Lemma 4.5 asserts that $\bar{T}_{j_2} \subset \bar{T}_{j_1}$, $h(e_{j_1}) \notin \bar{T}_{j_2}$.

Define $J_2 = \{j \in J: h(e_j) \in \bar{T}_{j_2}\}$, and so on. Since $\bar{T}_{j_0} \supset \bar{T}_{j_1} \supset \bar{T}_{j_2} \supset \ldots$, we must eventually find a j^* satisfying the conclusion of the lemma.

We show next that $\bar{T}_{j^*} \cap V_1 = \emptyset$. To this end we examine the set of edges $(\bar{T}_{j^*} \cap V_1, \bar{T}_{j^*} - V_1)$ in H'. Suppose first that $(\bar{T}_{j^*} \cap V_1, \bar{T}_{j^*} - V_1) \neq \emptyset$. In this case there is an edge e with $t(e) \in \bar{T}_{j^*} \cap V_1$, $h(e) \in \bar{T}_{j^*} - V_1$, and hence e is one of the edges e_j, $j \in J$. Since $h(e_j) \in \bar{T}_{j^*}$, Lemma 4.6 implies $\bar{T}_j = \bar{T}_{j^*}$. But we have $t(e_j) \notin \bar{T}_j$, $t(e_j) \in \bar{T}_{j^*}$, contradicting $\bar{T}_j = \bar{T}_{j^*}$. Thus $(\bar{T}_{j^*} \cap V_1, \bar{T}_{j^*} - V_1) = \emptyset$. But then $\bar{T}_{j^*} - V_1$ is regular with core $h(e_{j^*})$ and hence, since $\bar{T}_{j^*}$ is minimal with respect to this property, we must have $\bar{T}_{j^*} = \bar{T}_{j^*} - V_1$, which implies $\bar{T}_{j^*} \cap V_1 = \emptyset$.

Thus we have established the existence of $j^* \in J$ and $\bar{T}_{j^*}$ such that

$$(4.7) \qquad |\delta^-_{H'}(\bar{T}_{j^*})| = k - 1 \quad \text{and} \quad \bar{T}_{j^*} \cap V_1 = \emptyset.$$

It follows from (4.7) that

$$(4.8) \qquad |\delta^-_{H}(\bar{T}_{j^*})| = |\delta^-_{H'}(\bar{T}_{j^*})| = k - 1.$$

Thus $(T_{j^*}, \bar{T}_{j^*})$ is a cut in H separating s from $h(e_{j^*}) \in V$ having only $k - 1$ members, contradicting Lemma 4.1. Hence our assumption that (3.3) does not hold is untenable. This completes the proof of Theorem 3.2.

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